

Twisted cyclic cohomology and modular Fredholm modules

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Abstract

Connes and Cuntz showed in [CC] that suitable cyclic cocycles can be represented as Chern characters of finitely summable semifinite Fredholm modules. We show an analogous result in twisted cyclic cohomology using Chern characters of modular Fredholm modules. We present examples of modular Fredholm modules arising from Podleś spheres and from $SU_q(2)$.

1 Introduction

Let A be an associative algebra over the field of complex numbers k , $A * A$ the free product, and qA the ideal generated by $\iota(a) - \bar{\iota}(a)$, $a \in A$, where $\iota, \bar{\iota}$ are the two canonical inclusions of A in $A * A$. In [CC], it was shown that those cyclic cocycles for A which arise from positive traces on $(qA)^n$ are Chern characters of finitely summable semifinite Fredholm modules.

In this note we show that those twisted cyclic cocycles arising from KMS weights on $(qA)^n$ are Chern characters of finitely summable modular Fredholm modules, a twisted version of the usual notion of Fredholm modules. While this is not in any way a practical method of obtaining such representing Fredholm modules, it shows that in general one must consider the semifinite and modular settings.

The examples show that we can construct non-trivial twisted cyclic cocycles from naturally arising modular Fredholm modules. Moreover these cocycles encode the correct classical dimension, in the sense that the Hochschild class of these cocycles is non-vanishing at the classical dimension. Thus using twisted cohomology avoids the ‘dimension drop’ phenomena, at least in these examples.

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2 The algebraic background

We begin by a short recollection of the cyclic twisted cohomology of an algebra \mathcal{A} .

Definition 2.1. Let \mathcal{A} be an algebra and σ be an automorphism of \mathcal{A} . We say that $\phi : \mathcal{A}^{\otimes(n+1)} \rightarrow \mathbb{C}$ is a σ -twisted cyclic n -cocycle if for all $a_0, \dots, a_n \in \mathcal{A}$:

- ϕ is σ -invariant:

$$\phi(a_0, a_1, \dots, a_n) = \phi(\sigma(a_0), \sigma(a_1), \dots, \sigma(a_n));$$

- ϕ is σ -cyclic:

$$\phi(a_0, a_1, \dots, a_n) = (-1)^n \phi(\sigma(a_n), a_0, a_1, \dots, a_{n-1});$$

- ϕ is a σ -twisted Hochschild cocycle

$$\begin{aligned} (b^\sigma \phi)(a_0, a_1, \dots, a_n, a_{n+1}) &= \sum_{k=0}^n (-1)^k \phi(a_0, \dots, a_k a_{k+1}, \dots, a_{n+1}) \\ &\quad + (-1)^{n+1} \phi(\sigma(a_{n+1}) a_0, a_1, \dots, a_n) = 0. \end{aligned}$$

Let \mathcal{A} be an algebra and $q\mathcal{A}$ be its extension generated by \mathcal{A} and symbols $q(a)$, $a \in \mathcal{A}$, subject to the relation

$$q(ab) = q(a)b + aq(b) - q(a)q(b), \tag{2.1}$$

for all $a, b \in \mathcal{A}$. Equivalently, one may identify $q\mathcal{A}$ with the ideal within the free product algebra $\mathcal{A} * \mathcal{A}$ generated by the elements $q(a) := \iota(a) - \bar{\iota}(a)$ for $a \in \mathcal{A}$. If \mathcal{A} is an involutive algebra, then so too is $q\mathcal{A}$ with the involution defined by

$$q(a)^* = q(a^*), \quad a \in \mathcal{A}. \tag{2.2}$$

Setting $\mathcal{J} := q\mathcal{A} \subset \mathcal{A} * \mathcal{A}$, we can define \mathcal{J}^n to be the ideal of $\mathcal{A} * \mathcal{A}$ generated by the products $a_0 q(a_1) \cdots q(a_m)$ and $q(a_1) \cdots q(a_m)$ for $m \geq n$. If σ is an automorphism of \mathcal{A} , then we can extend σ to an automorphism of \mathcal{J} and \mathcal{J}^n by setting $\sigma(q(a)) := q(\sigma(a))$.

We present a simple generalisation of a result of Connes and Cuntz.

Proposition 2.1 (see [CC], Proposition 3). *Let \mathcal{A} be a unital algebra, σ an automorphism of \mathcal{A} , and let \mathcal{J} be the ideal $q\mathcal{A}$ of $\mathcal{A} * \mathcal{A}$ described above. Suppose that T is a σ -twisted trace on \mathcal{J}^n for some even integer n . That is, T is a linear functional such that*

$$T(xy) = T(\sigma(y)x), \quad \forall x \in \mathcal{J}^k, y \in \mathcal{J}^l, k + l = n. \quad (2.3)$$

Then the formula

$$\tau(a_0, a_1, \dots, a_n) := T(q(a_0)q(a_1) \cdots q(a_n)), \quad a_0, a_1, \dots, a_n \in \mathcal{A}$$

defines a σ -twisted cyclic n -cocycle τ on \mathcal{A} .

Proof. Since \mathcal{A} is unital, it follows immediately from Equation (2.3) that T is σ -invariant. It then follows that τ is σ -invariant and σ -cyclic. We verify explicitly that $b^\sigma \tau = 0$. So letting $a_0, \dots, a_n \in \mathcal{A}$ be arbitrary, we find that

$$\begin{aligned} (b^\sigma \phi)(a_0, a_1, \dots, a_n, a_{n+1}) &= \\ &= \sum_{k=0}^n (-1)^k T(q(a_0) \cdots q(a_k a_{k+1}) \cdots q(a_{n+1})) \\ &\quad + (-1)^{n+1} T(q(\sigma(a_{n+1})a_0)q(a_1) \cdots q(a_n)) \\ &= \sum_{k=0}^n (-1)^k T(q(a_0) \cdots (q(a_k)a_{k+1} + a_k q(a_{k+1}) - q(a_k)q(a_{k+1})) \cdots q(a_{n+1})) \\ &\quad + (-1)^{n+1} T((q(\sigma(a_{n+1}))a_0 + \sigma(a_{n+1})q(a_0) - q(\sigma(a_{n+1}))q(a_0))q(a_1) \cdots q(a_n)) \\ &= T(a_0 q(a_1) \cdots q(a_{n+1})) + T(q(a_0) \cdots q(a_n) a_{n+1}) - T(q(a_0) \cdots q(a_n) q(a_{n+1})) \\ &\quad - T((q(\sigma(a_{n+1}))a_0 - \sigma(a_{n+1})q(a_0) + q(\sigma(a_{n+1}))q(a_0))q(a_1) \cdots q(a_n)) \\ &= 0, \end{aligned}$$

where we used the twisted cyclicity of the functional T and the fact that n is even. \square

For the analogous statement for odd cocycles, we need to extend the automorphism σ to \mathcal{J}^n in a different way, [CC]. We define $\tilde{\sigma}$ via the formula

$$\begin{aligned} \tilde{\sigma}(a_0 q(a_1) \cdots q(a_m)) &= (-1)^m (\sigma(a_0) - q(\sigma(a_0))) q(\sigma(a_1)) \cdots q(\sigma(a_m)), \\ \tilde{\sigma}(q(a_1) \cdots q(a_m)) &= (-1)^m q(\sigma(a_1)) \cdots q(\sigma(a_m)). \end{aligned}$$

Then it is easy to check that $\tilde{\sigma}$ is indeed an automorphism of $q\mathcal{A}$ and, just as above, we have

Corollary 2.2. *If T is a $\tilde{\sigma}$ -twisted trace on \mathcal{J}^n , for n an odd integer, then the formula*

$$\tau(a_0, a_1, \dots, a_n) := T(q(a_0)q(a_1) \cdots q(a_n)), \quad a_0, a_1, \dots, a_n \in \mathcal{A},$$

defines a σ -twisted n -cyclic cocycle for \mathcal{A} .

3 The analytic picture

In this section we look at a version of [CC, Théorème 15] in twisted cyclic cohomology. In brief, [CC] shows that positive traces on certain ideals in the free product $\mathcal{A} * \mathcal{A}$ give rise to cyclic cocycles on \mathcal{A} . These cyclic cocycles can be represented as the Chern characters of semifinite Fredholm modules. By replacing traces with KMS functionals, we arrive at an analogue of this result in twisted cyclic theory. There are also some analytic differences in our starting assumptions, which we discuss at the end of this section.

We let A be a C^* -algebra and consider the free product C^* -algebra $A * A$. We denote by $\iota, \bar{\iota}$ the two canonical inclusions of A in $A * A$, and by qA the ideal generated by elements of the form $q(a) := \iota(a) - \bar{\iota}(a)$.

Similarly, if $\mathcal{A} \subset A$ is a dense subalgebra, then we let $q\mathcal{A}$ be the analogously defined ideal in $\mathcal{A} * \mathcal{A}$. Introduce the shorthand $J^k := (qA)^k$ and $\mathcal{J}^k := (q\mathcal{A})^k$ for $k \in \mathbb{N}$.

Our starting point is a faithful norm lower semicontinuous and norm semifinite KMS_β weight ϕ on J^{2p} with respect to a strongly continuous one parameter group $\sigma_\bullet : \mathbb{R} \rightarrow \text{Aut}(J^{2p})$. We will assume that $\mathcal{J}^{2p} \subset \text{dom}(\phi)$ and that \mathcal{J}^{2p} consists of analytic vectors for σ_\bullet .

The weight ϕ gives, via the GNS construction, a Hilbert space \mathcal{H}_ϕ with a nondegenerate representation $\pi_\phi : J^{2p} \rightarrow \mathcal{B}(\mathcal{H}_\phi)$, and a linear map $\Lambda : \text{dom}^{1/2} \phi \subset J^{2p} \rightarrow \mathcal{H}_\phi$. There is a canonical faithful normal semifinite extension Φ of ϕ to $(\pi_\phi(J^{2p}))''$ satisfying $\phi = \Phi \circ \pi_\phi$ and $\sigma_t^\Phi \circ \pi_\phi = \pi_\phi \circ \sigma_{-it}$. See [T, Proposition 1.5, Chapter VIII] for a proof.

The KMS property implies that for $a, b \in \mathcal{A}$ we have

$$\phi(ab) = \phi(\sigma(b)a),$$

where we define the (non-*) automorphism σ to be the value of the extension of the one-parameter group σ_\bullet to the complex value $t = i\beta$. Thus $\sigma := \sigma_{i\beta}$.

We observe that the representation of J^{2p} on \mathcal{H}_ϕ extends naturally to a representation of $A * A$ on \mathcal{H}_ϕ , denoted λ , such that $\lambda(A * A) \subset (\pi_\phi(J^{2p}))''$. This is the usual extension, defined on the dense subspace $\pi_\phi(J^{2p})\mathcal{H}_\phi$ by $\lambda(\alpha)(j\xi) := (\alpha j)\xi$ for $\alpha \in A * A$, $j \in J^{2p}$ and $\xi \in \mathcal{H}_\phi$. If T is in the commutant of $\pi_\phi(J^{2p})$ then

$$T(\lambda(\alpha)(j\xi)) = T((\alpha j)\xi)T = (\alpha j)(T\xi) = \lambda(\alpha)(j(T\xi)) = \lambda(\alpha)(T(j\xi)),$$

showing that $\lambda(\alpha)$ is indeed in $\pi_\phi(J^{2p})''$ for all $\alpha \in A * A$.

By [T, Theorem 2.6, Chapter VII], the (image under Λ of) $\text{dom}^{1/2} \Phi \cap (\text{dom}^{1/2} \Phi)^*$ is a full left Hilbert algebra, which we denote by U . Moreover, the left von Neumann algebra of U is precisely $(\pi_\phi(J^{2p}))''$. We record the following Lemma, whose proof follows immediately from the definitions.

Lemma 3.1. *Let N be the left von Neumann algebra of the left Hilbert algebra U and Φ the corresponding faithful normal semifinite weight. Then for all $\alpha \in J^p \cap \text{dom}^{1/2} \phi$ we have $\lambda(\alpha) \in \text{dom}^{1/2} \Phi$, and $\Phi(\lambda(\alpha)^* \lambda(\alpha)) = \phi(\alpha^* \alpha)$.*

Definition 3.1. Let \mathcal{N} be a von Neumann algebra acting on a Hilbert space \mathcal{H} . We fix a faithful normal strictly semifinite weight Φ . Then we say that $(\mathcal{A}, \mathcal{H}, F)$ is an n -summable **unital modular Fredholm module** with respect to (\mathcal{N}, Φ) if

- o) \mathcal{A} is a separable unital $*$ -subalgebra of \mathcal{N} ;
- i) \mathcal{A} is invariant under σ^Φ , and \mathcal{A} consists of analytic vectors for σ^Φ ;
- ii) F is a self-adjoint operator in fixed point algebra $\mathcal{M} := \mathcal{N}^{\sigma^\Phi}$ with $F^2 = 1_{\mathcal{N}}$;
- iii) $[F, a]$ extends to a bounded operator in \mathcal{N} for all $a \in \mathcal{A}$;
- iv) $[F, a]^n \in \text{dom } \Phi$ for all $a \in \mathcal{A}$.

The triple is even if there exists $\gamma = \gamma^* \in \mathcal{M}$ with $\gamma^2 = 1$, $\gamma a = a\gamma$ for all $a \in \mathcal{A}$ and $\gamma \mathcal{D} + \mathcal{D}\gamma = 0$. Otherwise the triple is odd.

The Chern character of a modular Fredholm module is the class of the $\sigma := \sigma_i^\Phi$ twisted cyclic n -cocycle defined by the formula (for suitable constants λ_n),

$$\text{Ch}_n(a_0, a_1, \dots, a_n) = \lambda_n \frac{1}{2} \Phi(\gamma F[F, a_0][F, a_1] \cdots [F, a_n]), \quad a_0, a_1, \dots, a_n \in \mathcal{A}.$$

Here we set $\gamma = 1$ if the module is odd.

Theorem 3.2. *Suppose that \mathcal{A} is a $*$ -subalgebra of a C^* -algebra A , and ϕ is a KMS_β weight on J^{2p} for the one parameter group σ such that $\mathcal{J}^{2p} = (q\mathcal{A})^{2p}$ consists of analytic vectors in the domain of ϕ . Then there exists a $2p$ -summable modular Fredholm module for \mathcal{A} . The modular Fredholm module has Chern character*

$$\text{Ch}_{2p}(a_0, a_1, \dots, a_{2p}) = \lambda_{2p} (-1)^p \phi(q(a_0) q(a_1) \cdots q(a_{2p})).$$

Proof. The universal property of $A * A$ gives two $*$ -homomorphisms $\pi_\phi, \bar{\pi}_\phi : A \rightarrow \mathcal{B}(L^2(J^{p+1}, \phi))$, whose images lie in $N = \pi_\phi(J^{2p})''$. The modular Fredholm module is given by the data:

- the Hilbert space $\mathcal{H} := L^2(J^{p+1}, \phi) \oplus L^2(J^{p+1}, \phi)$;
- the representation $\pi_2 : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$, $\pi_2(a) = \pi_\phi(a) \oplus \bar{\pi}_\phi(a)$;
- the operator $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;
- the von Neumann algebra $M_2(N)$;
- the weight $\Phi \circ \text{Tr}_2$.

Observe that

$$[F, \pi_2(a)] = \begin{pmatrix} 0 & \bar{\pi}_\phi(a) - \pi_\phi(a) \\ \pi_\phi(a) - \bar{\pi}_\phi(a) & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\pi_\phi(q(a)) \\ \pi_\phi(q(a)) & 0 \end{pmatrix}.$$

Since $J^{2p+1} \subset J^{2p}$, the Chern character is well-defined. The computation of the Chern character is straightforward. \square

Everything here is much simpler than in [CC] as we have assumed a little more with our starting data, in order to obtain a left Hilbert algebra. The reason for this is the necessity to prove the pre-closedness of the adjoint in the Hilbert space completion of our putative left Hilbert algebra. If we were to assume only a ‘positive twisted trace’ on \mathcal{J}^{2p} in some sense, we would not necessarily have enough control to prove this pre-closedness. By adding the full assumptions of a KMS weight, we get much more than we require, and everything follows from known results.

The odd case is similar using the $\tilde{\sigma}$ -automorphism of \mathcal{J}^n .

4 The modular index and paring with K-theory.

We recall here the construction of the modular index and its computation through the pairing between the equivariant K-theory and twisted cyclic cohomology. This section adapts [NT04] to the notation and notions used here.

Let $(\mathcal{A}, \mathcal{H}, F, \gamma)$ be an even modular Fredholm module with respect to (\mathcal{N}, Φ) (as defined above), which is $2n$ -summable. Assume that p is a projection, which is equivariant with respect to the automorphism σ_i^Φ of the algebra. If $p \in M_n(\mathcal{A})$ then we extend the modular Fredholm module to $(M_n(\mathcal{A}), \mathcal{H} \otimes \mathbb{C}^n, F \otimes \text{Id}, \gamma \otimes \text{Id})$ so that we can assume without loss of generality that $p \in \mathcal{A}$.

Furthermore, let us assume that there exists a densely defined operator Ξ in \mathcal{H} which implements the modular automorphism,

$$[F, \Xi] = 0, \quad [\Xi, \gamma] = 0, \quad \Xi^{-1}a\Xi = \sigma(a), \quad \forall a \in \mathcal{A},$$

where we identify a with the operator in the representation $\pi(\mathcal{A})$.

With this set up we can make the following definition, and we assume in what follows that $(\mathcal{N}, \Phi) = (\mathcal{B}(\mathcal{H}), \text{Tr}(\Xi^{1/2} \cdot \Xi^{1/2}))$, as this is the context we shall be working with in the examples. Extending the definition to the more general situation is straightforward using the theory of Breuer-Fredholm operators as in [CPRS3].

Definition 4.1. Let T be a Fredholm operator, which commutes with Ξ . We define the modular index of T to be

$$\text{q-Ind}(T) = \text{Tr}(\Xi|_{\ker T}) - \text{Tr}(\Xi|_{\text{coker } T}).$$

The definition makes sense, since both kernel and cokernel are finite-dimensional and at the same time invariant subspaces of K , so in fact both traces are finite expressions.

With this definition the following two propositions follow as in [NT04].

Proposition 4.1. *Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\gamma_{\mathcal{H}_{\pm}} = \pm \text{Id}$ and let us denote F_+ the restriction of F to \mathcal{H}_+ . For any σ equivariant projection $p \in \mathcal{A}$, the equivariant index of pF_+p is given by*

$$\text{q-Ind}(pF_+p) = (-1)^n \text{Tr}(\Xi \gamma p[F, p]^{2n}). \quad (4.1)$$

For the proof of the formula and its general properties (which follows the line of the proof for the standard Chern character) we refer to [NT04, Co].

For odd Fredholm modules, assume that we have an equivariant unitary element $U \in \mathcal{A}$, and set $E = \frac{1}{2}(1 + F)$. Then the equivariant index pairing is given by the modular index of $EUE : E\mathcal{H} \rightarrow E\mathcal{H}$.

Proposition 4.2.

$$\text{q-Ind}(EUE) = \frac{(-1)^n}{2^{2n}} \text{Tr}(\Xi ([F, U][F, U^{-1}])^n). \quad (4.2)$$

5 The Podleś spheres

5.1 The algebra

The family of Podleś quantum spheres $\mathcal{A}(S_{q,s}^2)$, $s \in [0, 1]$, is defined by generators A, B, B^* subject to the relations

$$\begin{aligned} B^*B + (A - 1)(A + s^2) &= 0, \\ BB^* + (q^2A - 1)(q^2A + s^2) &= 0, \\ AB &= q^{-2}BA. \end{aligned}$$

The two inequivalent irreducible representations of $\mathcal{A}(S_{q,s}^2)$ (for $s \in (0, 1]^1$) on $l^2(\mathbb{N})$, with standard orthonormal basis $\{e_k\}_{k \in \mathbb{N}}$, are given by the formulae,

$$\begin{aligned} \pi_+(B)e_k &= \sqrt{1 - q^{2k}} \sqrt{s^2 + q^{2k}} e_{k-1}, & \pi_+(A)e_k &= q^{2k} e_k, \\ \pi_-(B)e_k &= s \sqrt{1 - q^{2k}} \sqrt{1 + s^2 q^{2k}} e_{k-1}, & \pi_-(A)e_k &= -s^2 q^{2k} e_k. \end{aligned}$$

Definition 5.1. We construct an even Fredholm module $(\mathcal{A}(S_{q,s}^2), F, \mathcal{H})$ by taking $\mathcal{H} = l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$, the representation π defined for $a \in \mathcal{A}(S_{q,s}^2)$ by the formula

$$\pi(a) = \begin{pmatrix} \pi_+(a) & 0 \\ 0 & \pi_-(a) \end{pmatrix},$$

along with the grading operator and Fredholm operator F given by

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

¹The case $s = 0$ needs to be treated separately.

Lemma 5.1. *For $0 < s \leq 1$ and $0 < q < 1$, the Fredholm module $(\mathcal{A}(S_{q,s}^2), F, \mathcal{H})$ is a 2-summable modular Fredholm module, for the automorphism σ defined by*

$$\sigma(A) = A, \quad \sigma(B) = q^{-2}B, \quad \sigma(B^*) = q^2B^*.$$

The automorphism σ is implemented on the Hilbert space \mathcal{H} by the diagonal modular operator K

$$K e_{k,\pm} = q^{-2k} e_{k,\pm}, \quad \sigma(T) = K^{-1} T K, \quad T \in \pi(\mathcal{A}(S_{q,s}^2)).$$

The von Neumann algebra is $\mathcal{B}(\mathcal{H})$, and the weight Φ is defined by

$$\Phi(S) = \text{Tr}(K^{1/2} S K^{1/2}), \quad 0 \leq S \in \mathcal{B}(\mathcal{H}).$$

Proof. By [T, Theorem 2.11], Φ is a faithful normal semifinite weight on $\mathcal{B}(\mathcal{H})$, with modular group given by $T \mapsto K^{it} T K^{-it}$. Hence Φ is a σ -twisted trace for $\sigma(T) := \sigma_i^\Phi(T) := K^{-1} T K$.

As a next step we show that for any $a \in \mathcal{A}(S_{q,s}^2)$ the operator $K[F, \pi(a)]$ is bounded and $[F, a]$ is of trace class. Applying the definitions of the representation π and the operator F yields

$$\begin{aligned} K[F, \pi(A)] e_{k,\pm} &= \pm(1 + s^2) e_{k,\mp}, \\ K[F, \pi(B)] e_{k,\pm} &= \pm q^{-2k} \sqrt{1 - q^{2k}} \left(\sqrt{s^2 + q^{2k}} - s \sqrt{1 + s^2 q^{2k}} \right) e_{k-1,\mp}. \end{aligned}$$

To make an estimate for the last expression we observe that

$$\begin{aligned} \sqrt{s^2 + q^{2k}} - s \sqrt{1 + s^2 q^{2k}} &= \frac{(s^2 + q^{2k}) - s^2(1 + s^2 q^{2k})}{\sqrt{s^2 + q^{2k}} + s \sqrt{1 + s^2 q^{2k}}} \\ &= \frac{(1 - s^4) q^{2k}}{\sqrt{s^2 + q^{2k}} + s \sqrt{1 + s^2 q^{2k}}}, \end{aligned}$$

and since the denominator is greater than or equal to $2s$ we find

$$|\sqrt{s^2 + q^{2k}} - s \sqrt{1 + s^2 q^{2k}}| = \frac{(1 - s^4) q^{2k}}{\sqrt{s^2 + q^{2k}} + s \sqrt{1 + s^2 q^{2k}}} \leq \frac{1 - s^4}{2s} q^{2k}.$$

Therefore, in the end we obtain that for any $a_0, a_1 \in \mathcal{A}(S_{q,s}^2)$, the operator $K[F, a_0][F, a_1]$ is trace class, and so $[F, a_0][F, a_1]$ is in the domain of Φ . \square

Corollary 5.2. *The 3-linear functional ϕ defined by the formula*

$$\phi(a_0, a_1, a_2) = \Phi(\gamma F[F, a_0][F, a_1][F, a_2]), \quad a_0, a_1, a_2 \in \mathcal{A}(S_{q,s}^2),$$

determines a σ -twisted cyclic cocycle over $\mathcal{A}(S_{q,s}^2)$.

To see that the cyclic cocycle we obtained is non-trivial, we explicitly compute its pairing with the twisted cyclic cycle ω_2 , found by Hadfield [H]. In our notation the cyclic 2-cycle ω_2 is given by

$$\begin{aligned}\omega_2 = & 2(A \otimes B \otimes B^* - A \otimes B^* \otimes B + 2B \otimes B^* \otimes A - 2q^{-2}B \otimes A \otimes B^* \\ & + (q^4 - 1)A \otimes A \otimes A) + (1 - q^{-2})s^2(1 - s^2)1 \otimes 1 \otimes 1 \\ & + (1 - s^2)(1 \otimes B^* \otimes B - q^{-2}1 \otimes B \otimes B^* + (1 - q^2)1 \otimes A \otimes A)\end{aligned}$$

The pairing of ω_2 with ϕ (we skip the straightforward computations) gives

$$(\phi, \omega_2) = (1 + s^2)^3.$$

Since ω_2 comes from $HH_2^\sigma(\mathcal{A}(S_{q,s}^2))$, this also shows that the Hochschild class of ϕ is nontrivial.

5.2 The index pairing

The projection which, along with the class of 1, generates the σ -equivariant K_0 group of $\mathcal{A}(S_{q,s}^2)$, see [W], is given by

$$P = \frac{1}{1 + s^2} \begin{pmatrix} 1 - q^2 A & B \\ B^* & A + s^2 \end{pmatrix},$$

and the corresponding representation of the modular operator is

$$\rho(K) = \begin{pmatrix} \frac{1}{q} & 0 \\ 0 & q \end{pmatrix},$$

so that

$$\rho(K)P = \sigma(P)\rho(K).$$

We explicitly compute the index pairing of P with the twisted cocycle given by the Chern character of the modular Fredholm module constructed above. The pairing becomes

$$\begin{aligned}\langle \text{ch}(F, \pi, \mathcal{H}), P \rangle \\ = \frac{1}{2} \frac{2q}{(1 + s^2)^2} \sum_{k=0}^{\infty} (2s^2(q^4 - 1)q^{4k} + 2s(a_{k+1} - a_k) + (q^2 - 1)(1 - s^2)^2 q^{2k} + 2s(b_k - b_{k+1}))\end{aligned}$$

where

$$a_k = \sqrt{s^2 + q^{2k}} \sqrt{1 + s^2 q^{2k}} q^{2k} \quad \text{and} \quad b_k = \sqrt{s^2 + q^{2k}} \sqrt{1 + s^2 q^{2k}}.$$

We compute the sum explicitly. First, observe that

$$\sum_{k=0}^{\infty} (a_{k+1} - a_k) = -(1 + s^2), \quad \sum_{k=0}^{\infty} (b_k - b_{k+1}) = (1 + s^2) - s,$$

which then allows the rest of the sum to be computed to yield

$$\langle \text{ch}(F, \pi, \mathcal{H}), P \rangle = \frac{1}{2} \frac{2q}{(1+s^2)^2} (-2s^2 - 2s(1+s^2) - (1-s^2)^2 + 2s(1+s^2-s)) = -q.$$

Note that the index pairing is, of course, s -independent.

We now give an alternative picture of the index pairing using different generators for the equivariant K -theory group.

The algebra $\mathcal{A}(S_{q,s}^2)$ can be completed to a C^* -algebra $C(S_{q,s}^2)$ by means of the operator norm induced by the representation $\pi_+ \oplus \pi_-$. The modular group naturally extends by continuity to an action of $U(1)$ on $C(S_{q,s}^2)$. These $U(1)$ - C^* -algebras are known to be isomorphic to the fibre product of the two copies of the Toeplitz algebra \mathcal{T} with respect to the symbol map $\mathcal{T} \rightarrow C(S^1)$ [S]. Here, we consider the gauge action of $U(1)$ on each copy of \mathcal{T} and the translation action on $C(S^1)$.

Since $A \in \mathcal{A}(S_{q,s}^2)$ is σ -invariant, the spectral projections of the selfadjoint operator A give elements of the σ -equivariant K_0 group of $C(S_{q,s}^2)$. For each $k \in \mathbb{N}$, let us denote the projection onto the span of e_k in the representation space of π_+ (resp. of π_-) by $p_k^{(+)}$ (resp. by $p_k^{(-)}$). By the description of K in Lemma 5.1, we obtain the equalities

$$\langle \text{ch}(F, \pi, \mathcal{H}), p_k^{(\pm)} \rangle = \mp q^{-2k} \quad (k \in \mathbb{N}).$$

Next, let us relate the above to the computation above. Note that the presentation of P is operator norm continuous in the parameters $q \in [0, 1)$ and $s \in (0, 1]$. Hence the class of P in $K_0^{U(1)} C(S_{q,s}^2)$ is independent of q and s . Now, the projection $\pi_+ \oplus \pi_-(P)$ at $q = 0$ and $s = 1$ can be written as

$$\frac{1}{2} \begin{pmatrix} 1 & S^* \\ S & 1 + p_0^{(+)} \end{pmatrix} \oplus \frac{1}{2} \begin{pmatrix} 1 & S^* \\ S & 1 - p_0^{(-)} \end{pmatrix},$$

where S is the isometry $e_k \mapsto e_{k+1}$ on $l^2(\mathbb{N})$. This projection and

$$\begin{pmatrix} 1 & 0 \\ 0 & p_0^{(+)} \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

are connected by a continuous path of modular invariant projections $f_t^{(+)} \oplus f_t^{(-)}$ ($t \in [0, 1]$) defined by

$$f_t^{(+)} = \frac{1}{2} \begin{pmatrix} \sqrt{1-t^2} + 1 & tS^* \\ tS & (1 - \sqrt{1-t^2})(1 - p_0^{(+)} + 2p_0^{(+)}) \end{pmatrix}$$

and

$$f_t^{(-)} = \frac{1}{2} \begin{pmatrix} \sqrt{1-t^2} + 1 & tS^* \\ tS & (1 - \sqrt{1-t^2})(1 - p_0^{(-)}) \end{pmatrix}.$$

Hence we obtain

$$\langle \text{ch}(F, \pi, \mathcal{H}), P \rangle = \langle \text{ch}(F, \pi, \mathcal{H}), 1 \rangle q^{-1} + \langle \text{ch}(F, \pi, \mathcal{H}), p_0^{(+)} \rangle q = -q,$$

which gives a ‘global’ picture of the index pairing.

6 The modular Fredholm modules over $\mathcal{A}(SU_q(2))$

Let us recall the notation. The algebra $\mathcal{A}(SU_q(2))$ is generated by a, b satisfying the relations

$$ba = qab, \quad bb^* = b^*b, \quad b^*a = qab^*, \quad aa^* + bb^* = 1, \quad a^*a + q^2bb^* = 1.$$

In this section we shall demonstrate that the fundamental Fredholm module presented first in [MNW] and the Fredholm module arising from the spectral triple constructed in [DLSSV] both give rise to non-trivial twisted cyclic cocycles. We explicitly compute the pairing of these cocycles with an element from the equivariant K_1 group, and show that the two pairings are equal.

6.1 The basic Fredholm module

We briefly review the construction. The representation π_0 of the $\mathcal{A}(SU_q(2))$ algebra is given on the standard basis of $l^2(\mathbb{N}) \otimes l^2(\mathbb{Z})$ by

$$\begin{aligned} \pi_0(a)e_{k,l} &= \sqrt{1 - q^{2k+2}}e_{k+1,l}, \\ \pi_0(b)e_{k,l} &= q^ke_{k,l+1}, \end{aligned}$$

for $k \geq 0$, and $l \in \mathbb{Z}$. The Fredholm operator F is chosen to be

$$Fe_{k,l} = \text{sign}(l)e_{k,l}.$$

Lemma 6.1. *The triple (F, π_0, \mathcal{H}) is a 3-summable modular Fredholm module with respect to the von Neumann algebra $\mathcal{B}(\mathcal{H})$ and weight Φ defined as follows. Define the modular operator by*

$$Ke_{k,l} = q^{-2k}e_{k,l}.$$

Then the weight Φ is given by $\Phi(T) := \text{Tr}(K^{1/2}TK^{1/2})$, for $T \geq 0$.

Proof. We see that

$$[F, a] = 0, \quad [F, b]e_{k,l} = 2q^k\delta_{l,-1}e_{k,l+1}.$$

Therefore, for any two generators $x, y \in \{a, b\}$ of $SU_q(2)$ the operator $K[F, x][F, y]$ is bounded, and for any three $x, y, z \in \{a, b\}$, the operator $K[F, x][F, y][F, z]$ is trace class. \square

Since the Fredholm module is odd, we can use it to construct a twisted three-cyclic cocycle and pair the cocycle with the generator of the equivariant K_1 group, represented by the unitary V in $\mathcal{A}(SU_q(2)) \otimes M_2(\mathbb{C})$ given by

$$V = \begin{pmatrix} -qb^* & a \\ a^* & b \end{pmatrix}. \tag{6.1}$$

The twist arising from conjugation by the modular operator, $\sigma(T) = K^{-1}TK$, is given on algebra generators by

$$\sigma(b) = b, \quad \sigma(b^*) = b^*, \quad \sigma(a) = q^{-2}a, \quad \sigma(a^*) = q^2a^*.$$

If we extend the action of the modular operator on $\mathcal{H} \otimes \mathbb{C}^2$ via

$$\tilde{K} = K \otimes \begin{pmatrix} \frac{1}{q} & 0 \\ 0 & q \end{pmatrix},$$

then we see that V is invariant under conjugation by \tilde{K} , so

$$\tilde{\sigma}(V) = \tilde{K}^{-1}V\tilde{K} = V.$$

We can now compute the modular index pairing using (4.2).

Lemma 6.2. *The equivariant pairing between the class of Ch_F and the class of V is nontrivial and is equal to $-q$.*

Proof. We compute the pairing explicitly:

$$\begin{aligned} \langle [Ch_F], [V] \rangle &= \frac{(-1)^2}{2^4} \text{Tr} \left(\tilde{K} F[F, V][F, V^*][F, V][F, V^*] \right) \\ &= \frac{1}{16} \left(\sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}} 16q^{-2k} \text{sign}(l) (q^3 \delta_{l,0} q^{4k} + q \delta_{l,-1} q^{4k}) \right) \\ &= \sum_{k=0}^{\infty} q(q^2 - 1)q^{2k} = -q. \end{aligned} \quad \square$$

It follows from results of Hadfield and Krähmer (see [HK, Lemma 4.6]) that the map $I : HH_3^\sigma(\mathcal{A}(SU_q(2))) \rightarrow HC_3^\sigma(\mathcal{A}(SU_q(2)))$ is surjective. It then follows that the Hochschild class of the Chern character is nontrivial in $HH_\sigma^3(\mathcal{A}(SU_q(2)))$. Similar comments apply to the Chern character in the next section.

6.2 The modular Fredholm module of $SU_q(2)$ from its spectral triple.

The Fredholm module for $\mathcal{A}(SU_q(2))$ presented above gives (up to sign) the same (ordinary) K -homology class as the Fredholm module arising from the spectral triple over $\mathcal{A}(SU_q(2))$ discovered in [DLSSV]. It is therefore not surprising that the modular index pairings also agree, and that the twisted cyclic three-cocycles obtained from these two examples are both nontrivial.

Let us briefly recall the construction of the equivariant spectral triple over $\mathcal{A}(SU_q(2))$ due to [DLSSV]. We will use the notation from that work, and refer there for more details. Let $L^2(SU_q(2))$ denote the GNS representation space with respect to the Haar state. We decompose the space $H = L^2(SU_q(2)) \otimes V_{\frac{1}{2}}$, where $V_{\frac{1}{2}} \simeq \mathbb{C}^2$ is the fundamental representation of $U_q(su(2))$, so that we can write

$$H = \oplus_{l \in \frac{1}{2}\mathbb{N}} (V_l \otimes V_{\frac{1}{2}}) \otimes V_l.$$

Put $V_l^\uparrow = V_{l-}$ and $V_l^\downarrow = V_{l+}$. Then we have the decomposition

$$V_l \otimes V_{\frac{1}{2}} = V_l^\uparrow \oplus V_l^\downarrow.$$

If we put $W_l^\uparrow = V_l \otimes V_{l+}$ and $W_l^\downarrow = V_l \otimes V_{l-}$, we have a decomposition

$$(V_l \otimes V_{\frac{1}{2}}) \otimes V_l \simeq W_l^\uparrow \oplus W_l^\downarrow.$$

Setting $j^\pm = j \pm 1/2$, we have an orthonormal basis $|j\mu n \uparrow\rangle$ of W_j^\uparrow for $\mu = -j, -j+1, \dots, j$ and $n = -j^+, \dots, j^+$ (resp. a basis $|j\mu n \downarrow\rangle$ of W_j^\downarrow for $\mu = -j, -j+1, \dots, j$ and $n = -j^-, \dots, j^-$). For details, see [DLSSV].

The action of $\mathcal{A}(SU_q(2))$ on H is given ([DLSSV], Proposition 4.4) as follows. One may express the action of a as

$$\begin{aligned} \pi'(a) |j\mu n \uparrow\rangle &= \alpha_{j\mu n \uparrow \uparrow}^+ |j^+ \mu^+ n^+ \uparrow\rangle + \alpha_{j\mu n \uparrow \downarrow}^+ |j^+ \mu^+ n^+ \downarrow\rangle + \alpha_{j\mu n \uparrow \uparrow}^- |j^- \mu^+ n^+ \uparrow\rangle, \\ \pi'(a) |j\mu n \downarrow\rangle &= \alpha_{j\mu n \downarrow \downarrow}^+ |j^+ \mu^+ n^+ \downarrow\rangle + \alpha_{j\mu n \downarrow \uparrow}^- |j^- \mu^+ n^+ \downarrow\rangle + \alpha_{j\mu n \downarrow \downarrow}^- |j^- \mu^+ n^+ \uparrow\rangle, \end{aligned}$$

where the coefficients $\alpha_{j\mu n}^\pm$ are given by (writing $[k] = (q^{-k} - q^k)(q^{-1} - q)^{-1}$)

$$\begin{pmatrix} \alpha_{j\mu n \uparrow \uparrow}^+ & \alpha_{j\mu n \uparrow \downarrow}^+ \\ \alpha_{j\mu n \downarrow \uparrow}^+ & \alpha_{j\mu n \downarrow \downarrow}^+ \end{pmatrix} = q^{(\mu+n-\frac{1}{2})/2} [j + \mu + 1]^{\frac{1}{2}} \begin{pmatrix} q^{-j-\frac{1}{2}} \frac{[j+n+\frac{3}{2}]^{\frac{1}{2}}}{[2j+2]} & 0 \\ q^{\frac{1}{2}} \frac{[j-n+\frac{1}{2}]^{\frac{1}{2}}}{[2j+1][2j+2]} & q^{-j} \frac{[j+n+\frac{1}{2}]^{\frac{1}{2}}}{[2j+1]} \end{pmatrix}$$

and

$$\begin{pmatrix} \alpha_{j\mu n \uparrow \uparrow}^- & \alpha_{j\mu n \uparrow \downarrow}^- \\ \alpha_{j\mu n \downarrow \uparrow}^- & \alpha_{j\mu n \downarrow \downarrow}^- \end{pmatrix} = q^{(\mu+n-\frac{1}{2})/2} [j - \mu]^{\frac{1}{2}} \begin{pmatrix} q^{j+1} \frac{[j-n+\frac{1}{2}]^{\frac{1}{2}}}{[2j+1]} & -q^{\frac{1}{2}} \frac{[j+n+\frac{1}{2}]^{\frac{1}{2}}}{[2j][2j+1]} \\ 0 & q^{j+\frac{1}{2}} \frac{[j-n-\frac{1}{2}]^{\frac{1}{2}}}{[2j]} \end{pmatrix}.$$

Similarly, the action of b can be expressed as

$$\begin{aligned} \pi'(b) |j\mu n \uparrow\rangle &= \beta_{j\mu n \uparrow \uparrow}^+ |j^+ \mu^+ n^- \uparrow\rangle + \beta_{j\mu n \uparrow \downarrow}^+ |j^+ \mu^+ n^- \downarrow\rangle + \beta_{j\mu n \uparrow \uparrow}^- |j^- \mu^+ n^- \uparrow\rangle, \\ \pi'(b) |j\mu n \downarrow\rangle &= \beta_{j\mu n \downarrow \downarrow}^+ |j^+ \mu^+ n^- \downarrow\rangle + \beta_{j\mu n \downarrow \uparrow}^- |j^- \mu^+ n^- \downarrow\rangle + \beta_{j\mu n \downarrow \downarrow}^- |j^- \mu^+ n^- \uparrow\rangle, \end{aligned}$$

where the coefficients are given by

$$\begin{pmatrix} \beta_{j\mu n \uparrow \uparrow}^+ & \beta_{j\mu n \uparrow \downarrow}^+ \\ \beta_{j\mu n \downarrow \uparrow}^+ & \beta_{j\mu n \downarrow \downarrow}^+ \end{pmatrix} = q^{(\mu+n-\frac{1}{2})/2} [j + \mu + 1]^{\frac{1}{2}} \begin{pmatrix} \frac{[j-n+\frac{3}{2}]^{\frac{1}{2}}}{[2j+2]} & 0 \\ -q^{-j-1} \frac{[j+n+\frac{1}{2}]^{\frac{1}{2}}}{[2j+1][2j+2]} & q^{-\frac{1}{2}} \frac{[j-n+\frac{1}{2}]^{\frac{1}{2}}}{[2j+1]} \end{pmatrix}$$

and

$$\begin{pmatrix} \beta_{j\mu n \uparrow \uparrow}^- & \beta_{j\mu n \uparrow \downarrow}^- \\ \beta_{j\mu n \downarrow \uparrow}^- & \beta_{j\mu n \downarrow \downarrow}^- \end{pmatrix} = q^{(\mu+n-\frac{1}{2})/2} [j - \mu]^{\frac{1}{2}} \begin{pmatrix} -q^{-\frac{1}{2}} \frac{[j+n+\frac{1}{2}]^{\frac{1}{2}}}{[2j+1]} & -q^j \frac{[j-n+\frac{1}{2}]^{\frac{1}{2}}}{[2j][2j+1]} \\ 0 & -\frac{[j+n-\frac{1}{2}]^{\frac{1}{2}}}{[2j]} \end{pmatrix}.$$

The Dirac operator D acts as the scalar $-l$ on W_{l-}^\uparrow and as l on W_{l+}^\downarrow . The phase F of D is therefore given by the factor -1 on W_{l-}^\uparrow and by 1 on W_{l+}^\downarrow . In this basis, the modular element K is represented by

$$K |j\mu n \uparrow\downarrow\rangle = q^{-2(\mu+n)} |j\mu n \uparrow\downarrow\rangle.$$

Defining $\Phi(T) := \text{Tr}(K^{1/2}TK^{1/2})$ for $0 \leq T \in \mathcal{B}(H)$ gives us the weight we require.

Proposition 6.3. *The triple $(\mathcal{A}(SU_q(2)), H, F)$ is an odd 3-summable modular Fredholm module with respect to $(\mathcal{B}(H), \Phi)$.*

Proof. Since $x \mapsto [F, x]$ is a derivation, we only need to verify the summability condition for the generators $x = a, b$. Let P^\uparrow (resp. P^\downarrow) denote the projection onto $\oplus_j W_l^\uparrow$ (resp. $\oplus_j W_l^\downarrow$). Then the commutator $[F, x]$ can be expressed as $P^\uparrow x P^\downarrow - P^\downarrow x P^\uparrow$. Thus, for example,

$$[F, a] |j\mu n \uparrow\rangle \mapsto q^{(\mu+n-\frac{1}{2})/2} [j+\mu+1]^{\frac{1}{2}} q^{\frac{1}{2}} \frac{[j-n+\frac{1}{2}]^{\frac{1}{2}}}{[2j+1][2j+2]} |j^+\mu^+n^+\downarrow\rangle \quad (6.2)$$

$$[F, a] |j\mu n \downarrow\rangle \mapsto q^{(\mu+n-\frac{1}{2})/2} [j-\mu]^{\frac{1}{2}} q^{\frac{1}{2}} \frac{[j+n+\frac{1}{2}]^{\frac{1}{2}}}{[2j][2j+1]} |j^-\mu^+n^+\uparrow\rangle. \quad (6.3)$$

Therefore, we need to establish that the coefficients in the above expressions are summable with respect to the modular weight. The asymptotics of $[k]$ is the same as that of q^{-k} when $k \rightarrow \infty$. Hence the asymptotics of the first component of $[F, a]K^{\frac{1}{3}}$ is bounded from above by

$$\frac{q^{-(j+\frac{2}{3}\mu+\frac{1}{3}n)}}{q^{-4j}},$$

Similarly, from (6.3), the second component of $[F, a]K^{\frac{1}{3}}$ is asymptotically bounded from above by

$$\frac{q^{-(j+\frac{1}{3}\mu-\frac{2}{3}n)}}{q^{-4j}},$$

and one can see that it is a trace class operator. Analogously for $x = b$, using the expression of the matrices $\beta_{j\mu n}^\pm$, the ‘matrix coefficients’ of $[F, b]K^{\frac{1}{3}}$ are asymptotically bounded from above by

$$\frac{q^{-(2j+\frac{2}{3}\mu\frac{2}{3}n)}}{q^{-4j}}, \quad \frac{q^{-(j+\frac{1}{3}\mu-\frac{1}{3}n)}}{q^{-4j}},$$

and similar analysis shows that $[F, b]K^{\frac{1}{3}} \in L^1(H) \subset L^3(H)$. This proves the assertion. Observe that the Fredholm module $(\mathcal{A}(SU_q(2)), H, F)$ is not 2-summable, as could be easily seen by computing the asymptotics of $[F, b]K^{\frac{1}{2}}$, which shows that this is only bounded but not compact. \square

Since the product of at least two commutators with F is in the domain of the modular weight $\Phi(\cdot) = \text{Tr}(K^{1/2} \cdot K^{1/2})$ we can define the twisted Chern character of the modular Fredholm module as before:

$$\text{Ch}_{F,K}(x_0, x_1, x_2, x_3) = \frac{1}{2} \text{Tr}(F[F, x_0][F, x_1][F, x_2][F, x_3]K) \in \text{HC}_\theta^3(\mathcal{A}(SU_q(2))).$$

We now compute the pairing of $\text{Ch}_{F,K}$ with the equivariant odd K -group. Taking V as in (6.1) and with a similar extension of K to $H \otimes \mathbb{C}^2$, we obtain the following Proposition.

Proposition 6.4. *The modular index of V relative to the above Fredholm module is equal to 1.*

Proof. First of all, observe that V can be written using the fundamental unitary U of $\mathcal{A}(SU_q(2))$ via

$$V = SU, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} a & b \\ -qb^* & a^* \end{pmatrix}.$$

Recall that by [SDLSV], the operator PUP , where $P = \frac{1}{2}(1 + F) \otimes \text{Id}$, has a trivial cokernel, whereas its kernel is one dimensional and spanned by

$$\xi_0 = \begin{pmatrix} |0, 0, -\frac{1}{2} \uparrow \rangle \\ -q^{-1} |0, 0, \frac{1}{2} \uparrow \rangle \end{pmatrix}.$$

Since the matrix S commutes with the projection P , ξ_0 also spans the kernel of PVP . It is then easy to check that the eigenvalue of the modular operator \tilde{K} acting on ξ_0 is 1. \square

7 Conclusions

The significance of the results in [CC] is that we can represent those cyclic cocycles arising from traces on \mathcal{J}^n as Chern characters of n -summable semifinite Fredholm modules. Theorem 3.2 shows that we can represent those twisted cyclic cocycles arising from KMS weights on \mathcal{J}^n as Chern characters of modular Fredholm modules.

This Fredholm module approach to twisted traces works well, and in the examples avoids the dimension drop phenomena which plague q -deformations. An unbounded approach, in the spirit of spectral triples, is still a work in progress, but see [KS, KRS, RS].

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